Splitting schemes for the Navier-Stokes equations

In the notes/lectures it is stated that common splitting schemes like IPCS and variants can never be of higher order than 1. Explain why.

Incompressible Navier-Stokes equations

- $\rho\left(\frac{\partial u}{\partial t}+u\cdot\nabla u\right)=-\nabla p+\mu\nabla^2 u+\rho f$
- $\nabla\cdot u=0$
- Boundary conditions:
  - E. g. homogeneous Dirichlet: $u=0$ at Γ_D, p only needs to be specified **at a single point in space** (as a function of time)
  - E. g. homogeneous Neumann: $\mu\frac{\partial u}{\partial n}-p\cdot n\ =0$ at Γ_N (stress on the boundary)

Explicit scheme (Forward Euler)

- Forward Euler: $\left(\frac{\partial u}{\partial t}\right)^n \approx \frac{u^{n+1}-u^n}{\Delta t}$
- Solve for u^{n+1}: $\rho\left(\frac{u^{n+1}-u^n}{\Delta t} + u^n \cdot \nabla u^n\right) = -\nabla p^n + \mu\nabla^2 u^n + \rho f^n$
- Problems:
  - No natural way to compute p^{n+1}
  - No guarantee that $\nabla \cdot u^{n+1} = 0$
- What we want: $\rho\left(\frac{u^{n+1}-u^n}{\Delta t} + u^n \cdot \nabla u^n\right) = -\nabla p^{n+1} + \mu\nabla^2 u^n + \rho f^n$
  - Two unknowns, could fulfil $\nabla \cdot u^{n+1} = 0$

Explicit IPCS

- Solve for a tentative velocity:

$$\rho\left(\frac{u^* - u^n}{\Delta t} + u^n \cdot \nabla u^n\right) = -\nabla p^n + \mu \nabla^2 u^n + \rho f^n$$

- subtract: $\rho\left(\frac{u^{n+1} - u^n}{\Delta t} + u^n \cdot \nabla u^n\right) = -\nabla p^{n+1} + \mu \nabla^2 u^n + \rho f^n$
- Result: $\rho\left(\frac{u^{n+1} - u^*}{\Delta t}\right) = -\nabla(p^{n+1} - p^n)$
  - Can't be solved, as it has two unknowns
  - Can be used to derive boundary equations
  - Use $\nabla \cdot u^{n+1} = 0$
- Solve $\nabla \cdot \left(-\frac{\rho u^*}{\Delta t}\right) = -\nabla^2(p^{n+1} - p^n)$

Boundary conditions for the Poisson equation

$$\frac{\rho}{\Delta t} \nabla \cdot u^* = \nabla^2 (p^{n+1} - p^n) = \Delta \phi$$

- Boundary conditions for p are required **on the whole boundary**

1. Derive BCs from N-S equations
   - $\rho \left(\frac{\partial u}{\partial t} + u \cdot \nabla u \right) \cdot n = (-\nabla p + \mu \nabla^2 u + \rho f) \cdot n,$ on Γ
   - $\frac{\partial p}{\partial n} = \left(\mu \nabla^2 u + \rho f - \rho \frac{\partial u}{\partial t} - \rho u \cdot \nabla u \right) \cdot n,$ on Γ
   - Since $\phi = p^{n+1} - p^n$ and $p^{n+1} \neq p^n$ we obtain $\Delta \phi = \ldots \neq 0$ on Γ, a **non-homogenous condition**.

Boundary conditions for the Poisson equation

$$\frac{\rho}{\Delta t} \nabla \cdot u^* = \nabla^2 (p^{n+1} - p^n) = \Delta\phi$$

- Boundary conditions (BCs) for p are required **on the whole boundary**

2. Derive BCs from the scheme:
   - One can argue, that u^* has the proper Dirichlet boundary conditions already and u^{n+1} should have them too.
   - From $\rho\left(\frac{u^{n+1}-u^*}{\Delta t}\right) = -\nabla\phi$ follows, that $\Delta\phi = 0$ on Γ_D
   - The same reasoning applies for Neumann BCs: from $\rho\left(\frac{u^{n+1}-u^*}{\Delta t}\right) \cdot n = -\nabla\phi \cdot n$ follows that $\Delta\phi \cdot n = 0$ on Γ_N

Boundary conditions for the Poisson equation

$$\frac{\rho}{\Delta t} \nabla \cdot u^* = \nabla^2 (p^{n+1} - p^n) = \Delta\phi$$

- Boundary conditions for p are required **on the whole boundary**
- Different BCs are possible, the difference is first order
- Large error for the pressure near the boundary is expected
- An implicit IPCS is more stable, but involves the same Poisson equation
- We can set the right BCs for u^*. If we use homogenous Neumann conditions for $\Delta\phi$, we can assure that the normal component of the correction and thus of u^{n+1} is correct. However, this does not mean the tangential component is correct.